EXHIBIT B

1.3.6.6. Gallery of Distributions

Gallery of Common Distributions

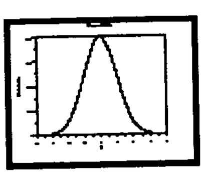
Detailed information on a few of the most common distributions is available below. There are a large number of distributions used in statistical applications. It is beyond the scope of this Handbook to discuss more than a few of these. Two excellent sources for additional detailed information on a large array of distributions are Johnson, Kotz, and Balakrishnan and Evans, Hastings, and Peacock.

Equations for the probability functions are given for the standard form of the distribution. Formulas exist for defining the functions with location and scale parameters in terms of the standard form of the distribution.

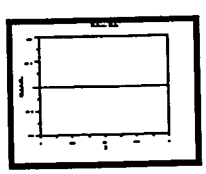
The sections on parameter estimation are restricted to the method of moments and maximum likelihood. This is because the least squares and PPCC and probability plot estimation procedures are generic. The maximum likelihood equations are not listed if they involve solving simultaneous equations. This is because these methods require sophisticated computer software to solve. Except where the maximum likelihood estimates are trivial, you should depend on a statistical software program to compute them. References are given for those who are interested.

Be aware that different sources may give formulas that are different from those shown here. In some cases, these are simply mathematically equivalent formulations. In other cases, a different parameterization may be used.

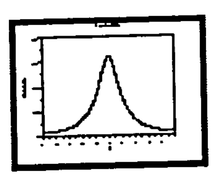
Continuous Distributions



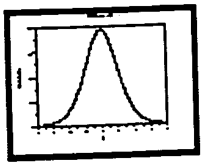
Normal Distribution



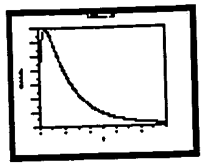
Uniform Distribution



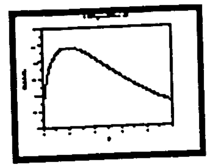
Cauchy Distribution



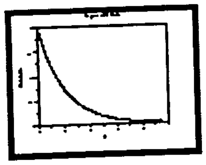
t Distribution



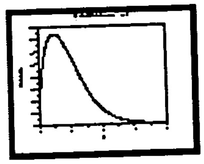
F Distribution



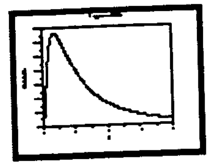
Chi-Square Distribution



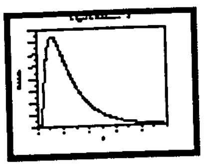
Exponential Distribution



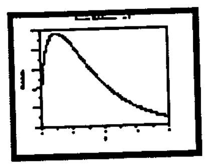
Weibull Distribution



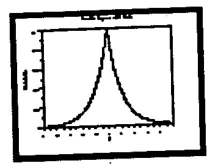
Lognormal Distribution



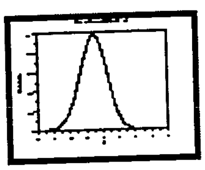
Fatigue Life Distribution



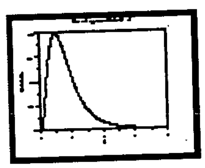
Gamma Distribution



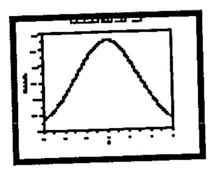
Double Exponential Distribution



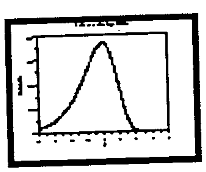
Power Normal Distribution



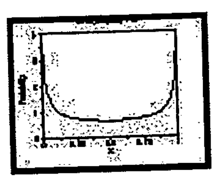
Power Lognormal Distribution



Tukey-Lambda Distribution

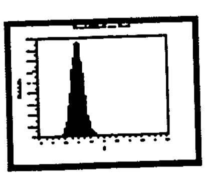


Extreme Value Type I Distribution

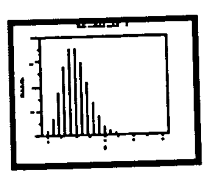


Beta Distribution

Discrete Distributions



Binomial Distribution



Poisson Distribution

1.3.6.6.1. Normal Distribution

Probability Density Function

The general formula for the probability density function of the normal distribution is

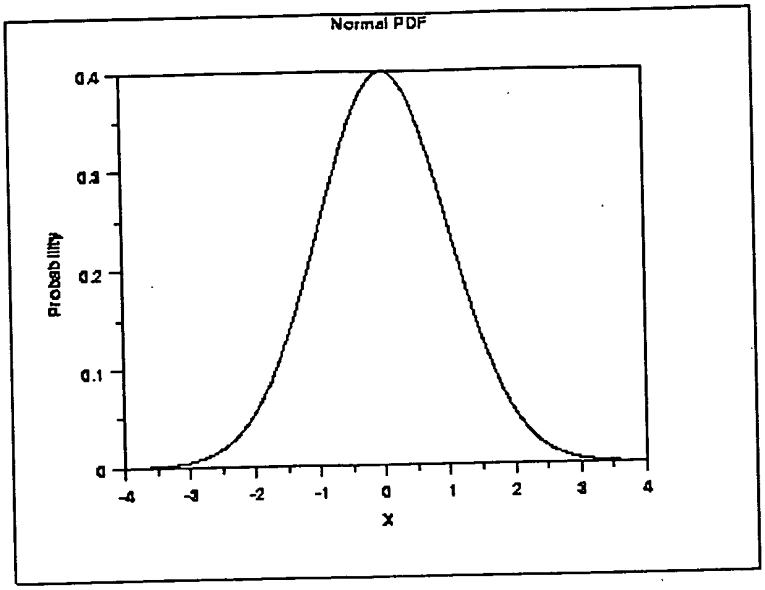
$$f(x) = \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}}$$

where μ is the location parameter and σ is the scale parameter. The case where $\mu = 0$ and $\sigma = 1$ is called the **standard normal distribution**. The equation for the standard normal distribution is

$$f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

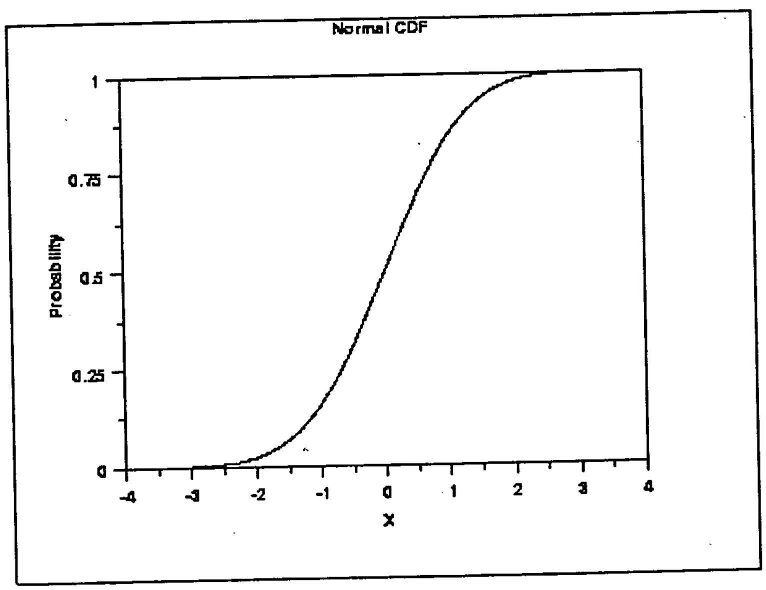
The following is the plot of the standard normal probability density function.



Cumulative Distribution Function

The formula for the cumulative distribution function of the normal distribution does not exist in a simple closed formula. It is computed numerically.

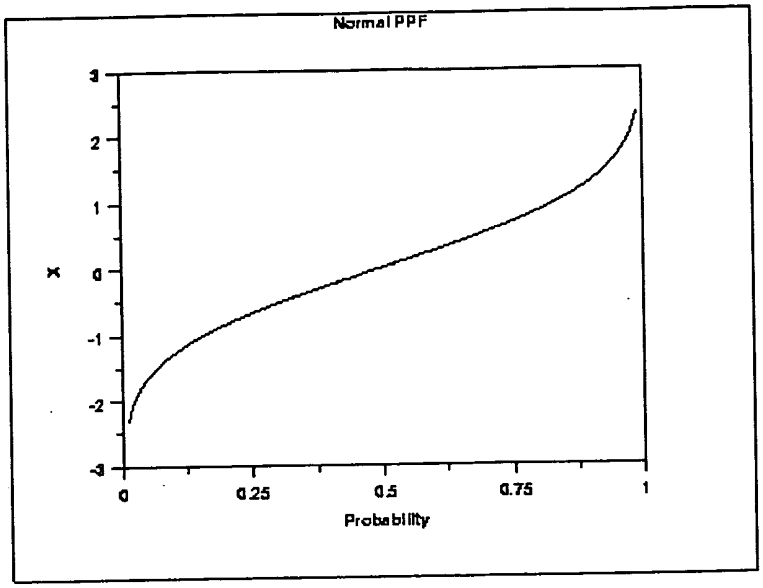
The following is the plot of the normal cumulative distribution function.



Percent Point Function

The formula for the percent point function of the normal distribution does not exist in a simple closed formula. It is computed numerically.

The following is the plot of the normal percent point function.



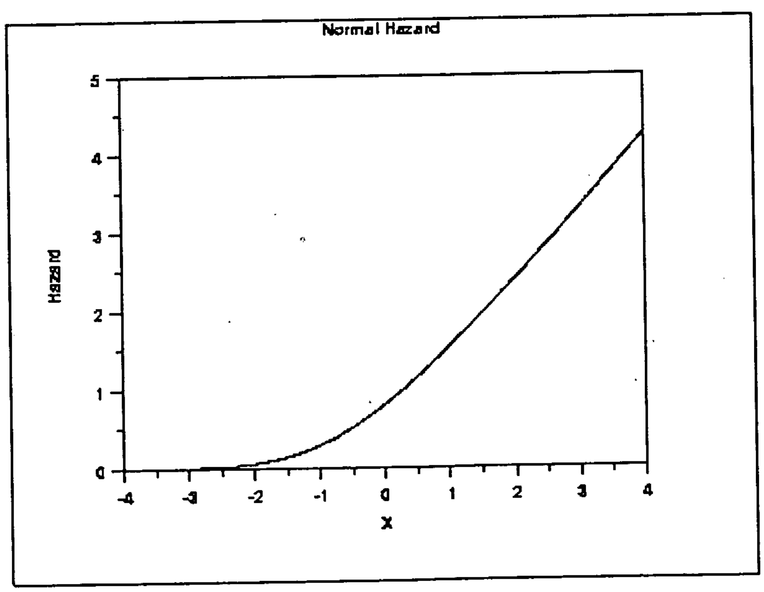
Hazard Function

The formula for the hazard function of the normal distribution is

$$h(x) = \frac{\phi(x)}{\Phi(-x)}$$

where Φ is the cumulative distribution function of the standard normal distribution and ϕ is the probability density function of the standard normal distribution.

The following is the plot of the normal hazard function.



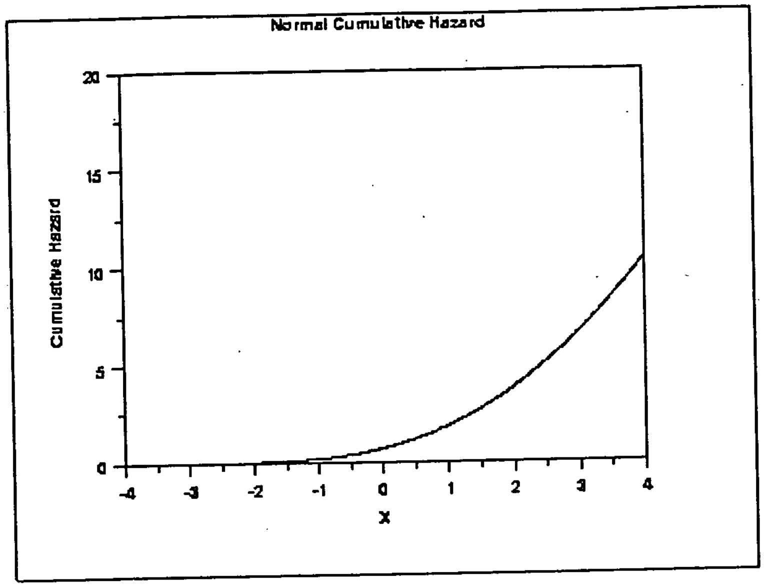
Cumulative

The normal cumulative hazard function can be computed

Hazard Function

from the normal cumulative distribution function.

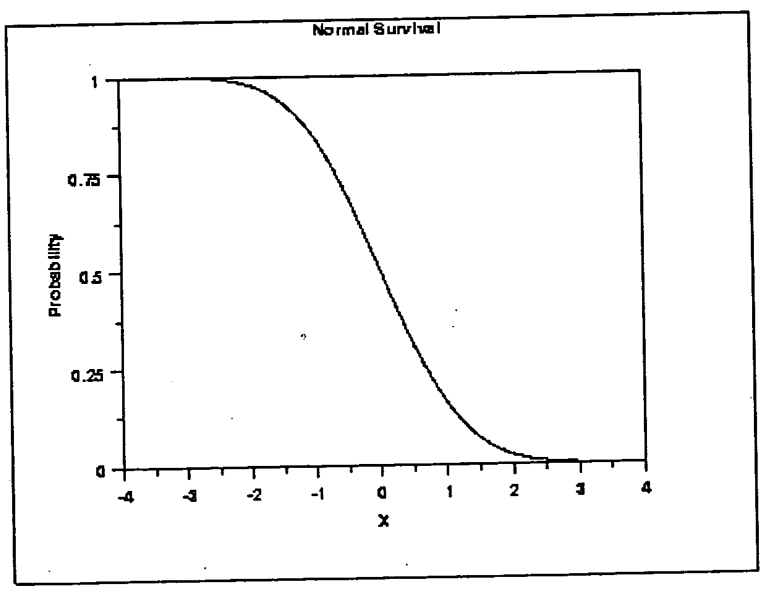
The following is the plot of the normal cumulative hazard function.



Survival Function

The normal survival function can be computed from the normal cumulative distribution function.

The following is the plot of the normal survival function.

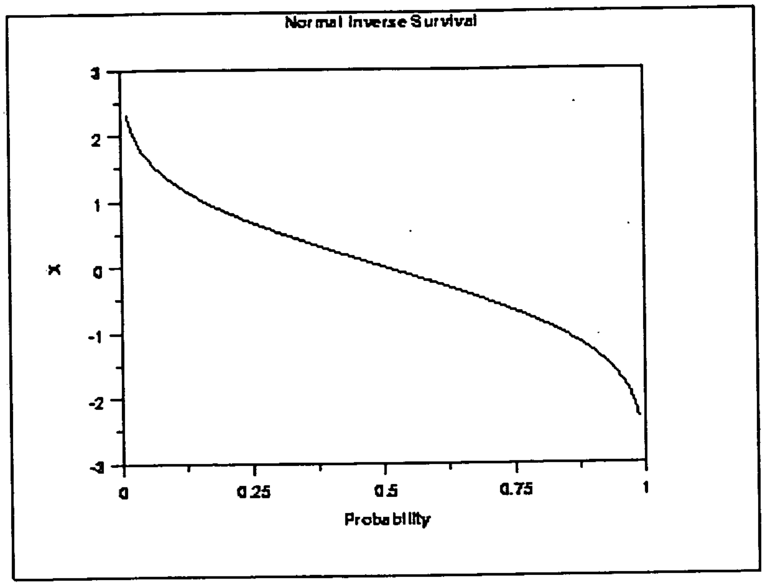


Inverse Survival Function

The normal inverse survival function can be computed from the normal percent point function.

The following is the plot of the normal inverse survival

function.



Common Statistics	Mean	The location parameter μ.
	Median	The location parameter μ.
	Mode	The location parameter μ.
	Range	Infinity in both directions.
	Standard Deviation	The scale parameter σ.
	Coefficient of Variation	σ/μ
	Skewness	0
	Kurtosis	3

Parameter Estimation

The location and scale parameters of the normal distribution can be estimated with the sample mean and sample standard deviation, respectively.

Comments

For both theoretical and practical reasons, the normal distribution is probably the most important distribution in statistics. For example,

- Many classical statistical tests are based on the assumption that the data follow a normal distribution. This assumption should be tested before applying these tests.
- In modeling applications, such as linear and non-linear regression, the error term is often assumed to follow a normal distribution with fixed location and scale.

- The normal distribution is used to find significance levels in many hypothesis tests and confidence intervals.

Theroretical Justification - Central Limit Theorem

The normal distribution is widely used. Part of the appeal is that it is well behaved and mathematically tractable. However, the central limit theorem provides a theoretical basis for why it has wide applicability.

The central limit theorem basically states that as the sample size (N) becomes large, the following occur:

1. The sampling distribution of the mean becomes approximately normal regardless of the distribution of the original variable.

2. The sampling distribution of the mean is centered at the population mean, μ, of the original variable. In addition, the standard deviation of the sampling distribution of the mean approaches $\sigma/\sqrt{N}$.

Software

Most general purpose statistical software programs, including Dataplot, support at least some of the probability functions for the normal distribution.

1.3.6.6.2. Uniform Distribution

Probability Density Function

The general formula for the probability density function of the uniform distribution is

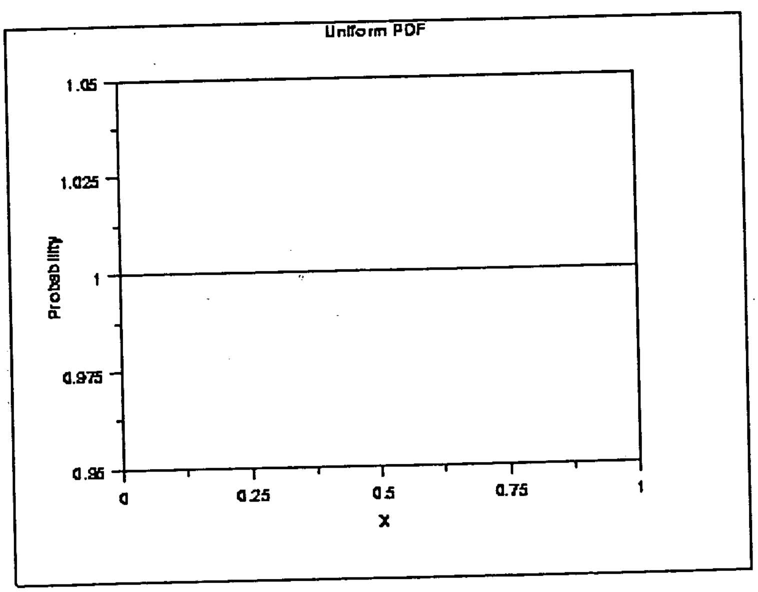
$$f(x) = \frac{1}{B-A} \qquad \text{for } A \le x \le B$$

where A is the location parameter and (B - A) is the scale parameter. The case where A = 0 and B = 1 is called the **standard uniform distribution**. The equation for the standard uniform distribution is

$$f(x) = 1 \qquad \text{for } 0 \le x \le 1$$

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the uniform probability density function.



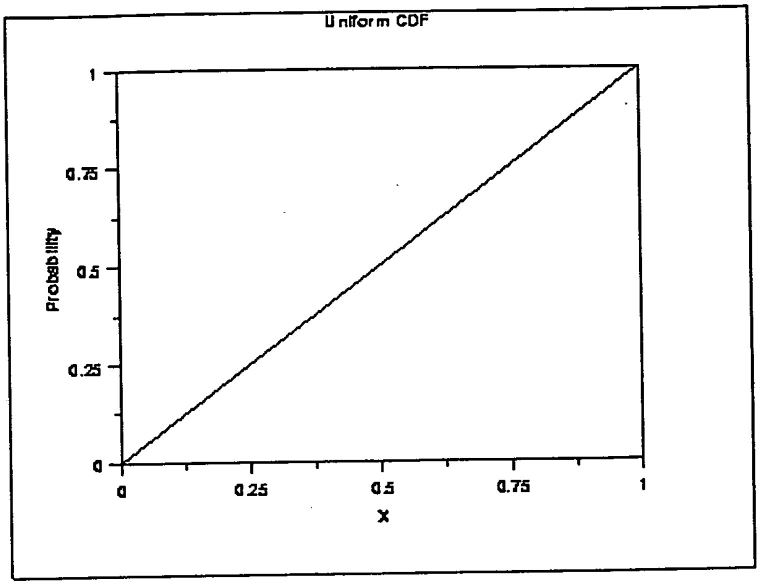
Cumulative Distribution

The formula for the cumulative distribution function of the uniform distribution is

Function

$$F(x) = x \quad \text{for } 0 \leq x \leq 1$$

The following is the plot of the uniform cumulative distribution function.

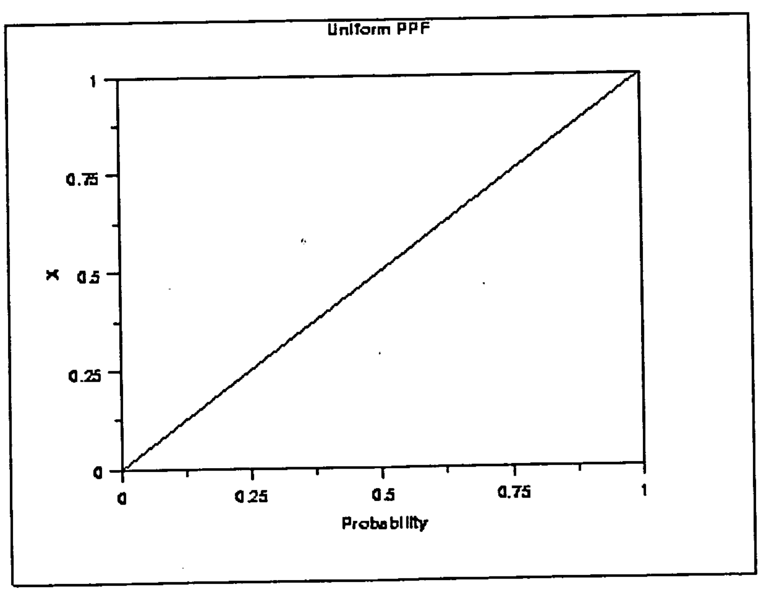


Percent Point Function

The formula for the percent point function of the uniform distribution is

$$G(p) = p \quad \text{for } 0 \leq p \leq 1$$

The following is the plot of the uniform percent point function.



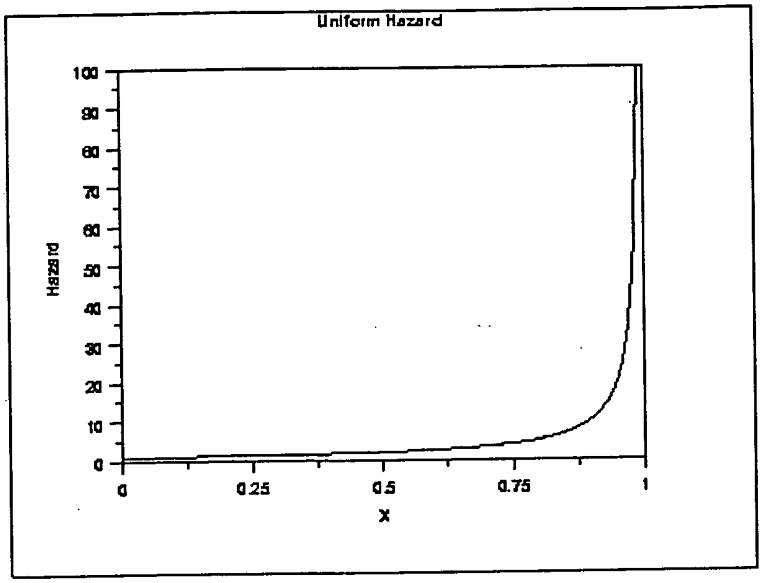
Hazard

The formula for the hazard function of the uniform distribution is

Function

$$h(x) = \frac{1}{1-x} \quad \text{for } 0 \le x < 1$$

The following is the plot of the uniform hazard function.

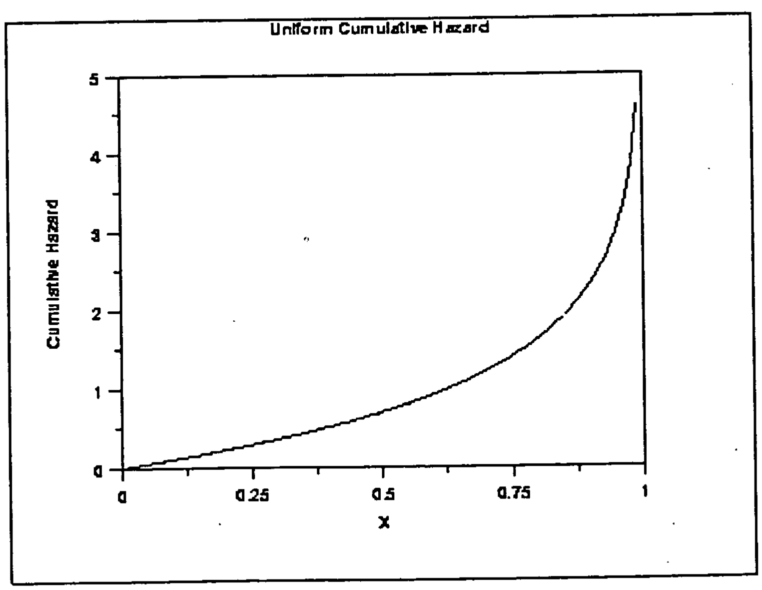


Cumulative Hazard Function

The formula for the cumulative hazard function of the uniform distribution is

$$H(x) = -ln(1-x) \quad \text{for } 0 \le x < 1$$

The following is the plot of the uniform cumulative hazard function.

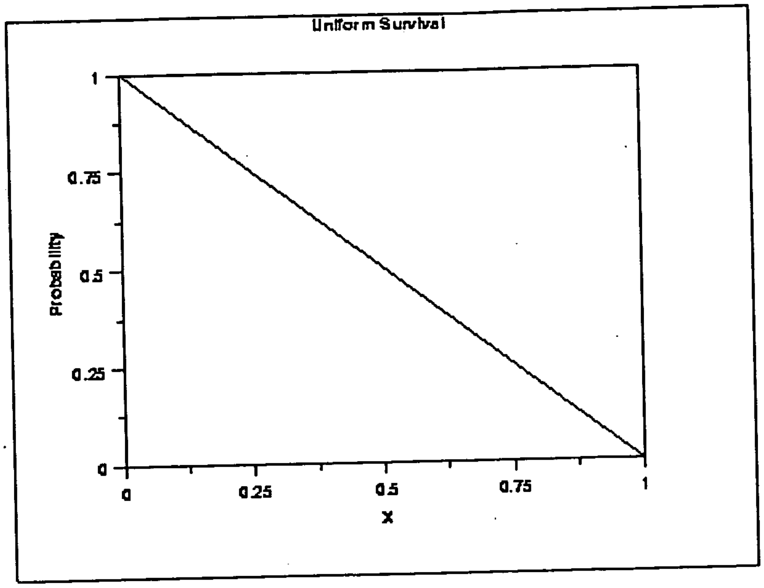


Survival

The uniform survival function can be computed from the uniform

Function cumulative distribution function.

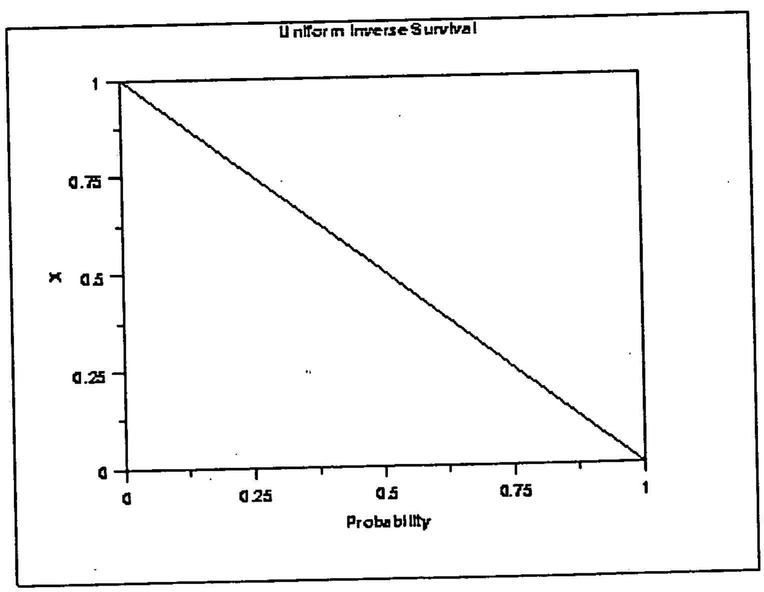
The following is the plot of the uniform survival function.



Inverse Survival Function

The uniform inverse survival function can be computed from the uniform percent point function.

The following is the plot of the uniform inverse survival function.



Common Statistics

Mean	(A + B)/2
Median	(A + B)/2
Range	B - A
Standard Deviation	

	$\sqrt{\frac{(B-A)^2}{12}}$
Coefficient of Variation	$\frac{(B-A)}{\sqrt{3}(B+A)}$
Skewness	0
Kurtosis	9/5

Parameter Estimation

The method of moments estimators for A and B are

$$A = \bar{x} - \sqrt{3}s$$
$$B = \bar{x} + \sqrt{3}s$$

The maximum likelihood estimators for A and B are

$$A = \text{midrange}(Y_1, Y_2, ..., Y_n) - 0.5[\text{range}(Y_1, Y_2, ..., Y_n]$$

$$B = \text{midrange}(Y_1, Y_2, ..., Y_n) + 0.5[\text{range}(Y_1, Y_2, ..., Y_n]$$

Comments

The uniform distribution defines equal probability over a given range for a continuous distribution. For this reason, it is important as a reference distribution.

One of the most important applications of the uniform distribution is in the generation of random numbers. That is, almost all random number generators generate random numbers on the (0,1) interval. For other distributions, some transformation is applied to the uniform random numbers.

Software

Most general purpose statistical software programs, including Dataplot, support at least some of the probability functions for the uniform distribution.

1.3.6.6.3. Cauchy Distribution

Probability Density Function

The general formula for the probability density function of the Cauchy distribution is

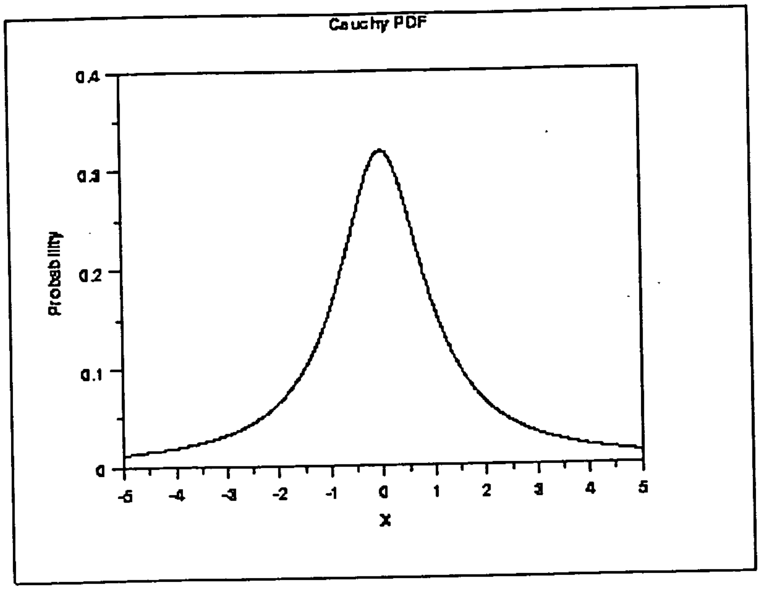
$$f(x) = \frac{1}{s\pi(1 + ((x - t)/s)^2)}$$

where t is the location parameter and s is the scale parameter. The case where $t = 0$ and $s = 1$ is called the **standard Cauchy distribution**. The equation for the standard Cauchy distribution reduces to

$$f(x) = \frac{1}{\pi(1 + x^2)}$$

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the standard Cauchy probability density function.

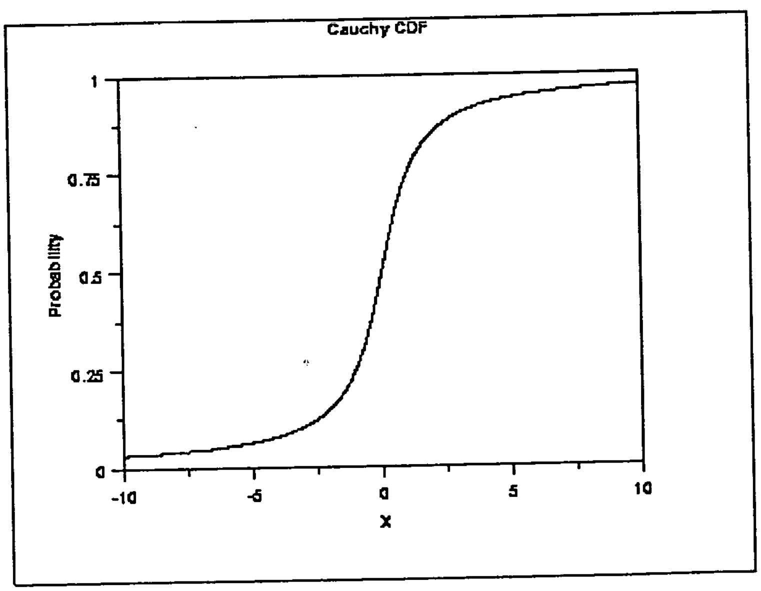


Cumulative Distribution Function

The formula for the cumulative distribution function for the Cauchy distribution is

$$F(x) = 0.5 + \frac{\arctan(x)}{\pi}$$

The following is the plot of the Cauchy cumulative distribution function.

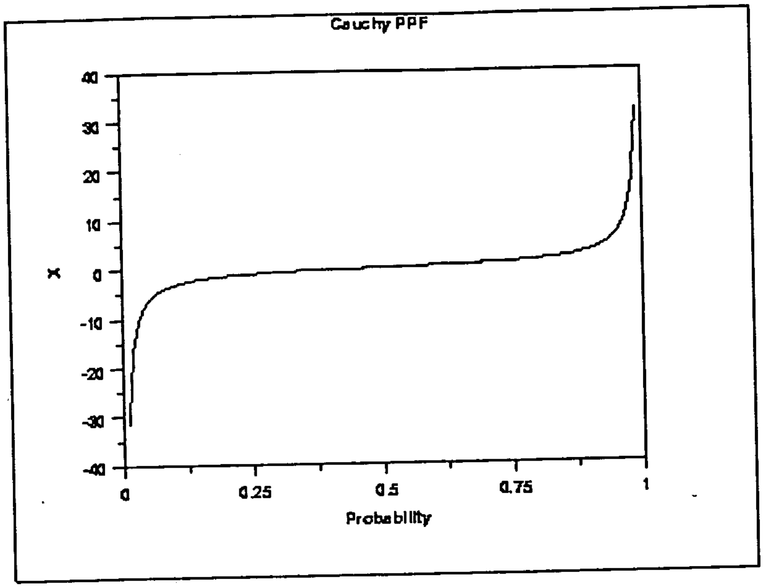


Percent Point Function

The formula for the percent point function of the Cauchy distribution is

$$G(p) = -\cot(\pi p)$$

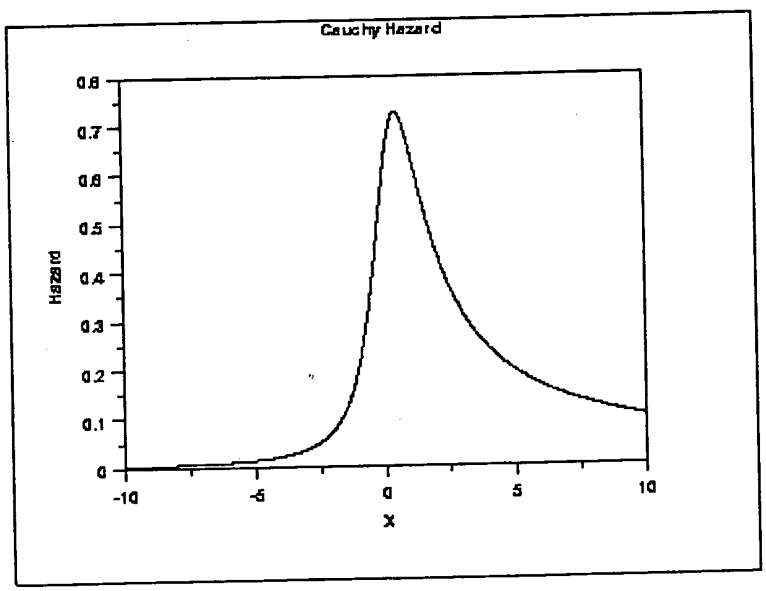
The following is the plot of the Cauchy percent point function.



Hazard Function

The Cauchy hazard function can be computed from the Cauchy probability density and cumulative distribution functions.

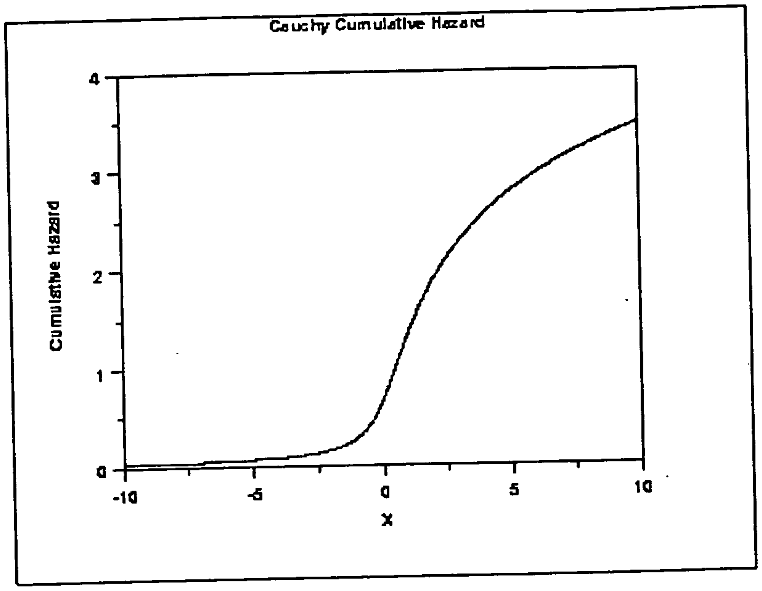
The following is the plot of the Cauchy hazard function.



Cumulative Hazard Function

The Cauchy cumulative hazard function can be computed from the Cauchy cumulative distribution function.

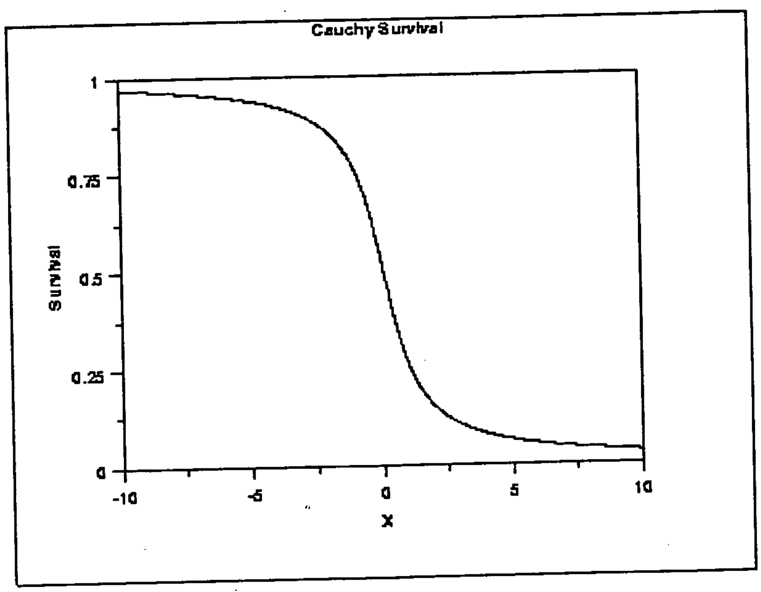
The following is the plot of the Cauchy cumulative hazard function.



Survival Function

The Cauchy survival function can be computed from the Cauchy cumulative distribution function.

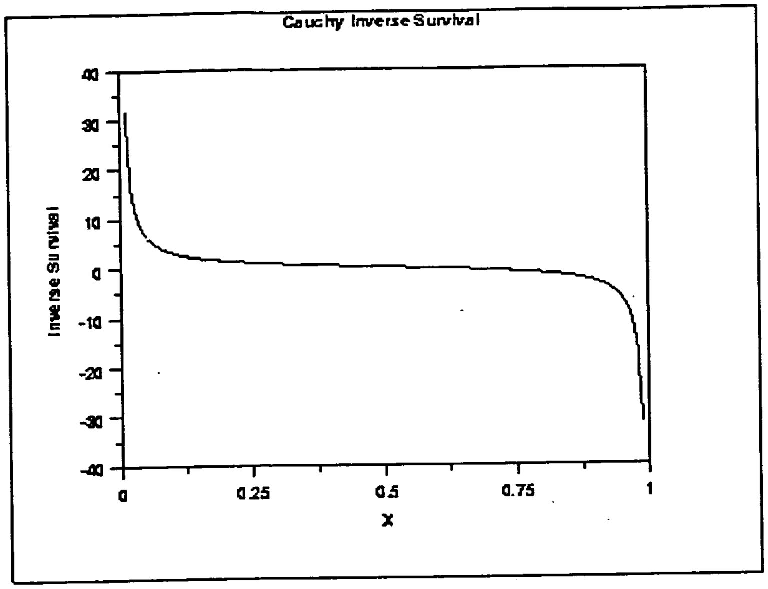
The following is the plot of the Cauchy survival function.



Inverse Survival Function

The Cauchy inverse survival function can be computed from the Cauchy percent point function.

The following is the plot of the Cauchy inverse survival function.



Common Statistics	Mean	The mean is undefined.
	Median	The location parameter *t*.
	Mode	The location parameter *t*.
	Range	Infinity in both directions.
	Standard Deviation	The standard deviation is undefined.
	Coefficient of Variation	The coefficient of variation is undefined.
	Skewness	The skewness is undefined.
	Kurtosis	The kurtosis is undefined.

Parameter Estimation

The likelihood functions for the Cauchy maximum likelihood estimates are given in chapter 16 of Johnson, Kotz, and Balakrishnan. These equations typically must be solved numerically on a computer.

Comments

The Cauchy distribution is important as an example of a pathological case. Cauchy distributions look similar to a normal distribution. However, they have much heavier tails. When studying hypothesis tests that assume normality, seeing how the tests perform on data from a Cauchy distribution is a good indicator of how sensitive the tests are to heavy-tail departures from normality. Likewise, it is a good check for robust techniques that are designed to work well under a wide variety of distributional assumptions.

The mean and standard deviation of the Cauchy distribution are undefined. The practical meaning of this is that collecting 1,000 data points gives no more accurate an estimate of the

mean and standard deviation than does a single point.

Software

Many general purpose statistical software programs, including Dataplot, support at least some of the probability functions for the Cauchy distribution.

1.3.6.6.4. t Distribution

Probability Density Function

The formula for the probability density function of the *t* distribution is

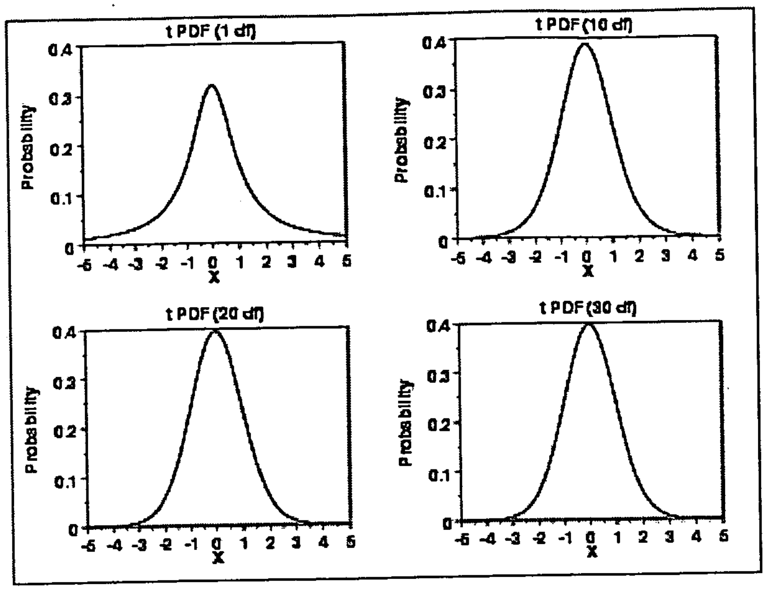
$$f(x) = \frac{\left(1+\frac{x^2}{\nu}\right)^{\frac{-(\nu+1)}{2}}}{B(0.5, 0.5\nu)\sqrt{\nu}}$$

where B is the beta function and ν is a positive integer shape parameter. The formula for the beta function is

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$$

In a testing context, the *t* distribution is treated as a "standardized distribution" (i.e., no location or scale parameters). However, in a distributional modeling context (as with other probability distributions), the *t* distribution itself can be transformed with a location parameter, μ, and a scale parameter, σ.

The following is the plot of the *t* probability density function for 4 different values of the shape parameter.

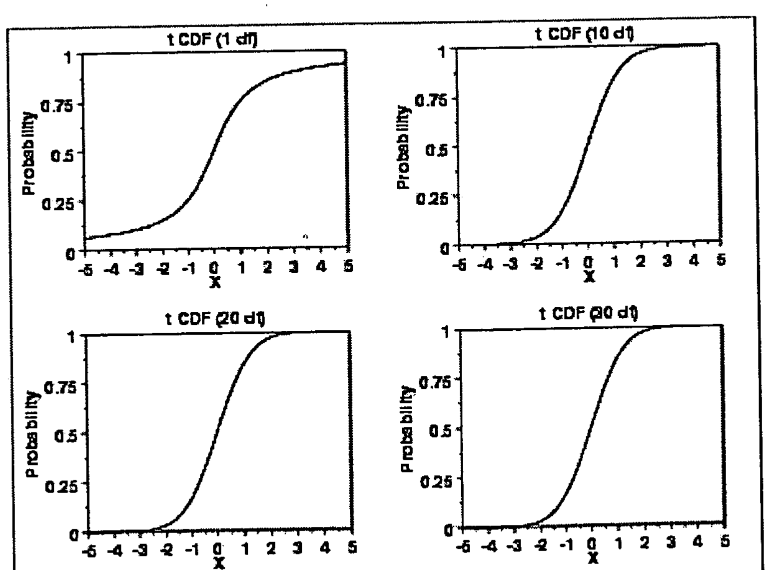


These plots all have a similar shape. The difference is in the heaviness of the tails. In fact, the t distribution with ν equal to 1 is a Cauchy distribution. The t distribution approaches a normal distribution as ν becomes large. The approximation is quite good for values of $\nu > 30$.

Cumulative Distribution Function

The formula for the cumulative distribution function of the t distribution is complicated and is not included here. It is given in the Evans, Hastings, and Peacock book.

The following are the plots of the t cumulative distribution function with the same values of ν as the pdf plots above.



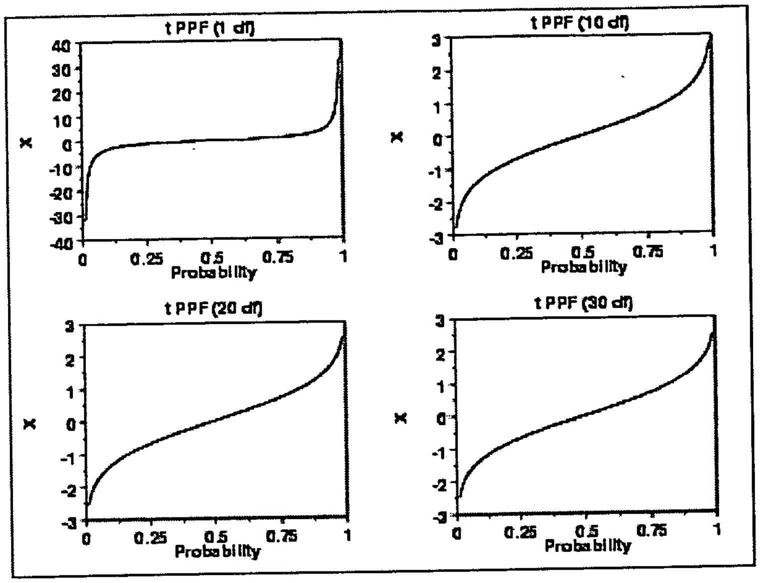
Percent

The formula for the percent point function of the t

Point Function

distribution does not exist in a simple closed form. It is computed numerically.

The following are the plots of the t percent point function with the same values of ν as the pdf plots above.



Other Probability Functions

Since the t distribution is typically used to develop hypothesis tests and confidence intervals and rarely for modeling applications, we omit the formulas and plots for the hazard, cumulative hazard, survival, and inverse survival probability functions.

Common Statistics

Mean	0 (It is undefined for ν equal to 1.)
Median	0
Mode	0
Range	Infinity in both directions.
Standard Deviation	$\sqrt{\frac{\nu}{(\nu-2)}}$ It is undefined for ν equal to 1 or 2.
Coefficient of Variation	Undefined
Skewness	0. It is undefined for ν less than or equal to 3. However, the t distribution is symmetric in all cases.
Kurtosis	$\frac{3(\nu-2)}{(\nu-4)}$

It is undefined for ν less than or equal to 4.

Parameter Estimation

Since the *t* distribution is typically used to develop hypothesis tests and confidence intervals and rarely for modeling applications, we omit any discussion of parameter estimation.

Comments

The *t* distribution is used in many cases for the critical regions for hypothesis tests and in determining confidence intervals. The most common example is testing if data are consistent with the assumed process mean.

Software

Most general purpose statistical software programs, including Dataplot, support at least some of the probability functions for the *t* distribution.

1.3.6.6.5. F Distribution

Probability Density Function

The F distribution is the ratio of two chi-square distributions with degrees of freedom ν_1 and ν_2, respectively, where each chi-square has first been divided by its degrees of freedom. The formula for the probability density function of the F distribution is

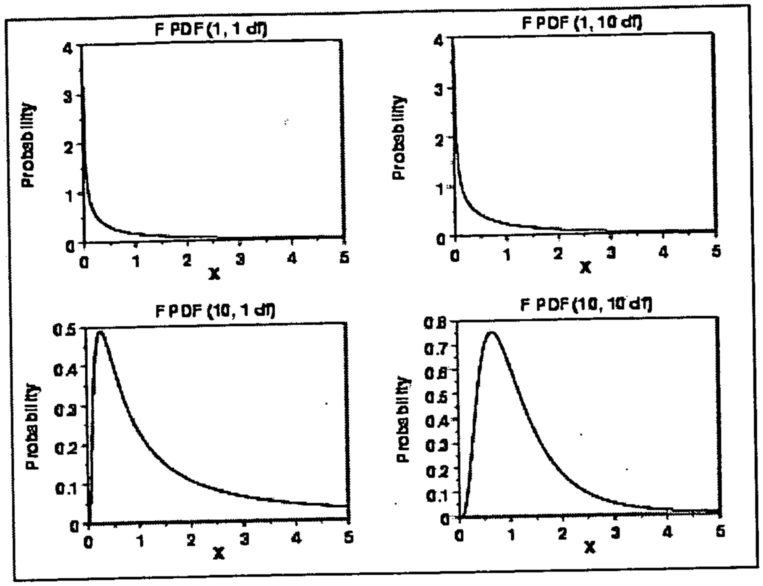
$$f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\frac{\nu_1}{\nu_2})^{\frac{\nu_1}{2}} x^{\frac{\nu_1}{2}-1}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})(1+\frac{\nu_1 x}{\nu_2})^{\frac{\nu_1+\nu_2}{2}}}$$

where ν_1 and ν_2 are the shape parameters and Γ is the gamma function. The formula for the gamma function is

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$$

In a testing context, the F distribution is treated as a "standardized distribution" (i.e., no location or scale parameters). However, in a distributional modeling context (as with other probability distributions), the F distribution itself can be transformed with a location parameter, μ, and a scale parameter, σ.

The following is the plot of the F probability density function for 4 different values of the shape parameters.



Cumulative Distribution Function

The formula for the Cumulative distribution function of the F distribution is

$$F(x) = 1 - I_k(\frac{\nu_2}{2}, \frac{\nu_1}{2})$$

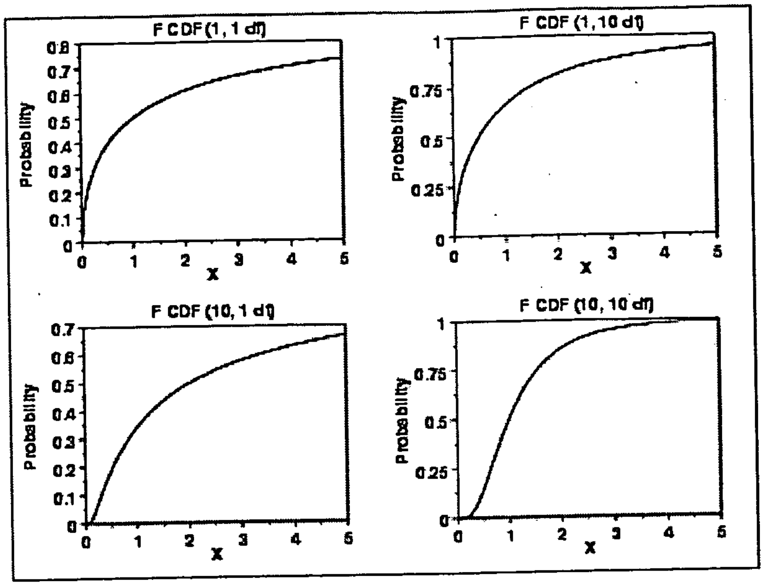
where k = ν_2/ (ν_2 + ν_1*x) and I_k is the incomplete beta function. The formula for the incomplete beta function is

$$I_k(x, \alpha, \beta) = \frac{\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt}{B(\alpha, \beta)}$$

where ***B*** is the beta function

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$$

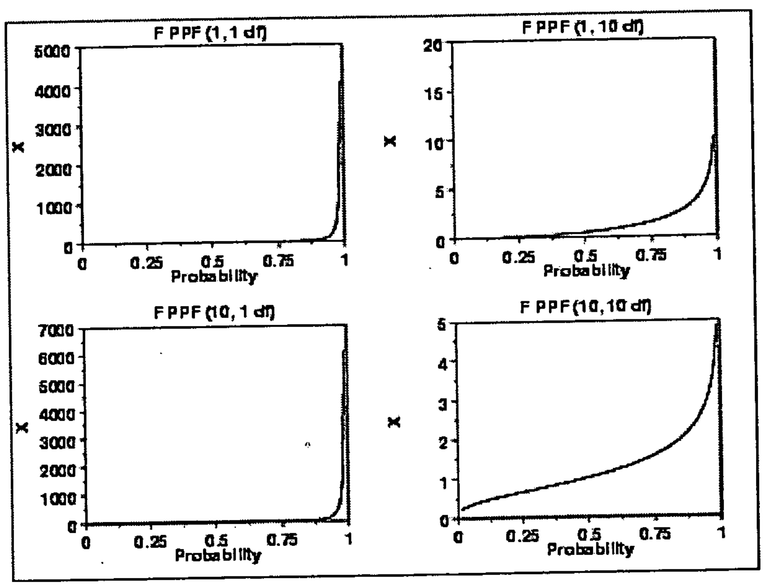
The following is the plot of the F cumulative distribution function with the same values of ν_1 and ν_2 as the pdf plots above.



Percent Point Function

The formula for the percent point function of the F distribution does not exist in a simple closed form. It is computed numerically.

The following is the plot of the F percent point function with the same values of ν_1 and ν_2 as the pdf plots above.



Other Probability Functions

Since the F distribution is typically used to develop hypothesis tests and confidence intervals and rarely for modeling applications, we omit the formulas and plots for the hazard, cumulative hazard, survival, and inverse survival probability functions.

Common

The formulas below are for the case where the location

Statistics parameter is zero and the scale parameter is one.

Mean	$\frac{\nu_2}{(\nu_2 - 2)}$	$\nu_2 > 2$
Mode	$\frac{\nu_2(\nu_1 - 2)}{\nu_1(\nu_2 + 2)}$	$\nu_1 > 2$
Range	0 to positive infinity	
Standard Deviation	$\sqrt{\frac{2\nu_2^2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)}}$	$\nu_2 > 4$
Coefficient of Variation	$\sqrt{\frac{2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 4)}}$	$\nu_2 > 4$
Skewness	$\frac{(2\nu_1 + \nu_2 - 2)\sqrt{8(\nu_2 - 4)}}{\sqrt{\nu_1}(\nu_2 - 6)\sqrt{(\nu_1 + \nu_2 - 2)}}$	$\nu_2 > 6$

Parameter Estimation Since the F distribution is typically used to develop hypothesis tests and confidence intervals and rarely for modeling applications, we omit any discussion of parameter estimation.

Comments The F distribution is used in many cases for the critical regions for hypothesis tests and in determining confidence intervals. Two common examples are the analysis of variance and the F test to determine if the variances of two populations are equal.

Software Most general purpose statistical software programs, including Dataplot, support at least some of the probability functions for the F distribution.

1.3.6.6.6. Chi-Square Distribution

Probability Density Function

The chi-square distribution results when ν independent variables with standard normal distributions are squared and summed. The formula for the probability density function of the chi-square distribution is

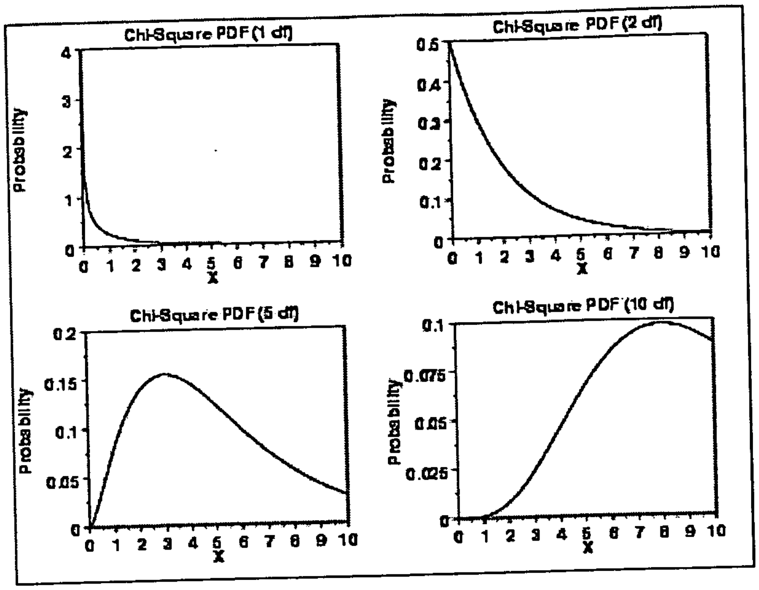
$$f(x) = \frac{e^{\frac{-x}{2}} x^{\frac{\nu}{2}-1}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \qquad \text{for } x \geq 0$$

where ν is the shape parameter and Γ is the gamma function. The formula for the gamma function is

$$\Gamma(a) = \int_0^{\infty} t^{a-1}e^{-t}dt$$

In a testing context, the chi-square distribution is treated as a "standardized distribution" (i.e., no location or scale parameters). However, in a distributional modeling context (as with other probability distributions), the chi-square distribution itself can be transformed with a location parameter, μ, and a scale parameter, σ.

The following is the plot of the chi-square probability density function for 4 different values of the shape parameter.



Cumulative Distribution Function

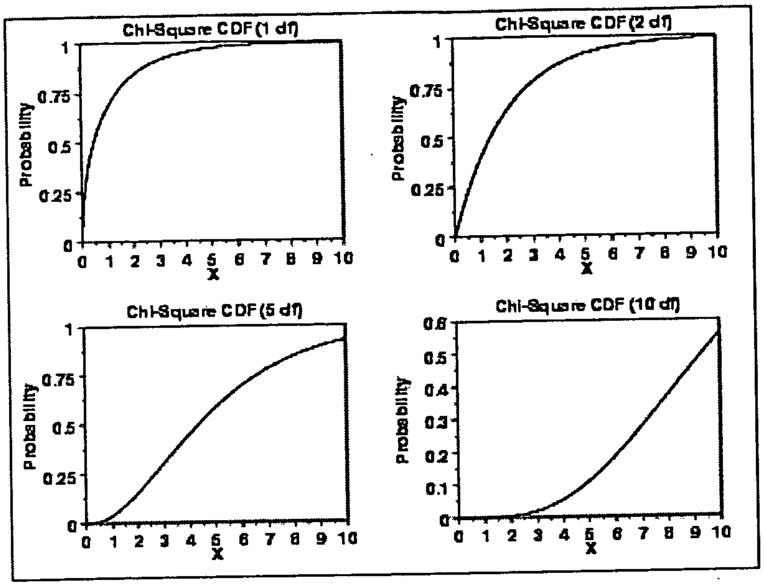
The formula for the cumulative distribution function of the chi-square distribution is

$$F(x) = \frac{\gamma(\frac{\nu}{2}, \frac{x}{2})}{\Gamma(\frac{\nu}{2})} \quad \text{for } x \ge 0$$

where Γ is the gamma function defined above and γ is the incomplete gamma function. The formula for the incomplete gamma function is

$$\Gamma_x(a) = \int_0^x t^{a-1} e^{-t} dt$$

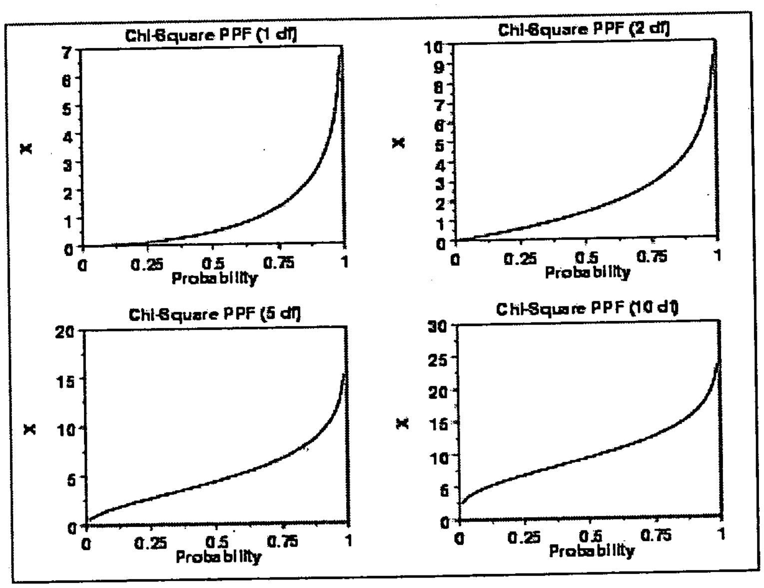
The following is the plot of the chi-square cumulative distribution function with the same values of ν as the pdf plots above.



Percent Point Function

The formula for the percent point function of the chi-square distribution does not exist in a simple closed form. It is computed numerically.

The following is the plot of the chi-square percent point function with the same values of ν as the pdf plots above.



Other Probability Functions

Since the chi-square distribution is typically used to develop hypothesis tests and confidence intervals and rarely for modeling applications, we omit the formulas and plots for the hazard, cumulative hazard, survival, and inverse survival probability functions.

Common

Mean ν

Statistics	Median	approximately ν - 2/3 for large ν
	Mode	$\nu - 2$ for $\nu > 2$
	Range	0 to positive infinity
	Standard Deviation	$\sqrt{2\nu}$
	Coefficient of Variation	$\sqrt{\frac{2}{\nu}}$
	Skewness	$\frac{2^{1.5}}{\sqrt{\nu}}$
	Kurtosis	$3+\frac{12}{\nu}$

Parameter Estimation

Since the chi-square distribution is typically used to develop hypothesis tests and confidence intervals and rarely for modeling applications, we omit any discussion of parameter estimation.

Comments

The chi-square distribution is used in many cases for the critical regions for hypothesis tests and in determining confidence intervals. Two common examples are the chi-square test for independence in an ***RxC*** contingency table and the chi-square test to determine if the standard deviation of a population is equal to a pre-specified value.

Software

Most general purpose statistical software programs, including Dataplot, support at least some of the probability functions for the chi-square distribution.

1.3.6.6.7. Exponential Distribution

Probability Density Function

The general formula for the probability density function of the exponential distribution is

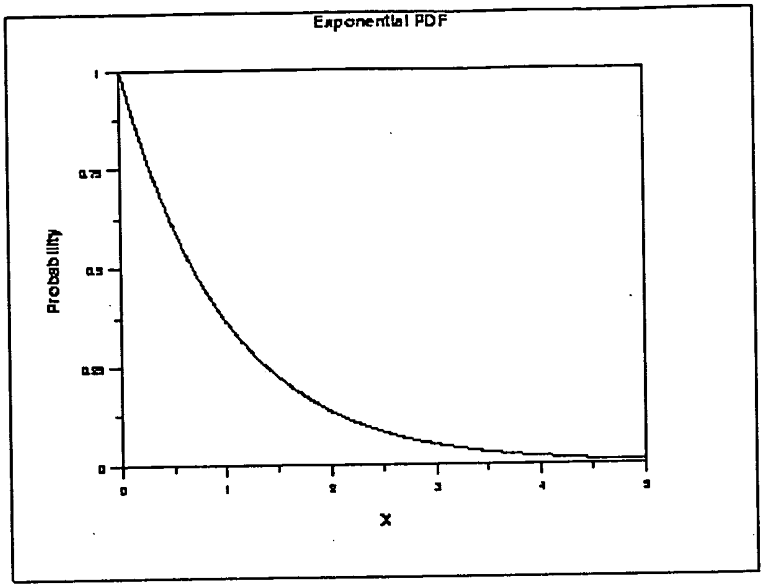
$$f(x) = \frac{1}{\beta}e^{-(x-\mu)/\beta} \qquad x \ge \mu; \beta > 0$$

where μ is the location parameter and β is the scale parameter (the scale parameter is often referred to as λ which equals $1/\beta$). The case where $\mu = 0$ and $\beta = 1$ is called the **standard exponential distribution**. The equation for the standard exponential distribution is

$$f(x) = e^{-x} \qquad \text{for } x \ge 0$$

The general form of probability functions can be expressed in terms of the standard distribution. Subsequent formulas in this section are given for the 1-parameter (i.e., with scale parameter) form of the function.

The following is the plot of the exponential probability density function.

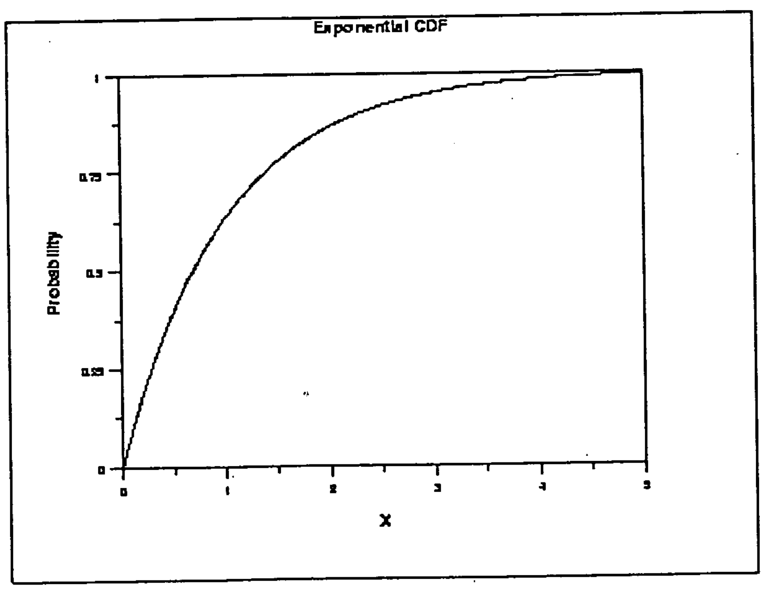


Cumulative Distribution Function

The formula for the cumulative distribution function of the exponential distribution is

$$F(x) = 1 - e^{-x/\beta} \qquad x \ge 0; \beta > 0$$

The following is the plot of the exponential cumulative distribution function.

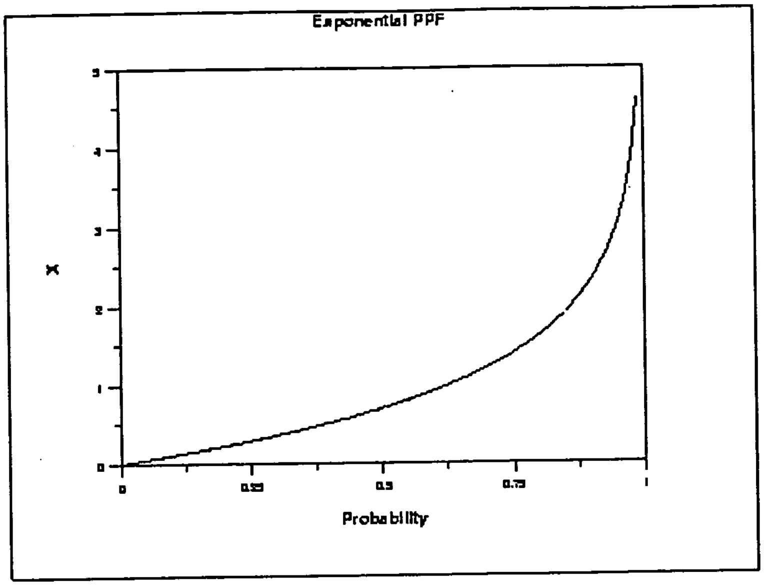


Percent Point Function

The formula for the percent point function of the exponential distribution is

$$G(p) = -\beta \ln(1 - p) \qquad 0 \le p < 1; \beta > 0$$

The following is the plot of the exponential percent point function.

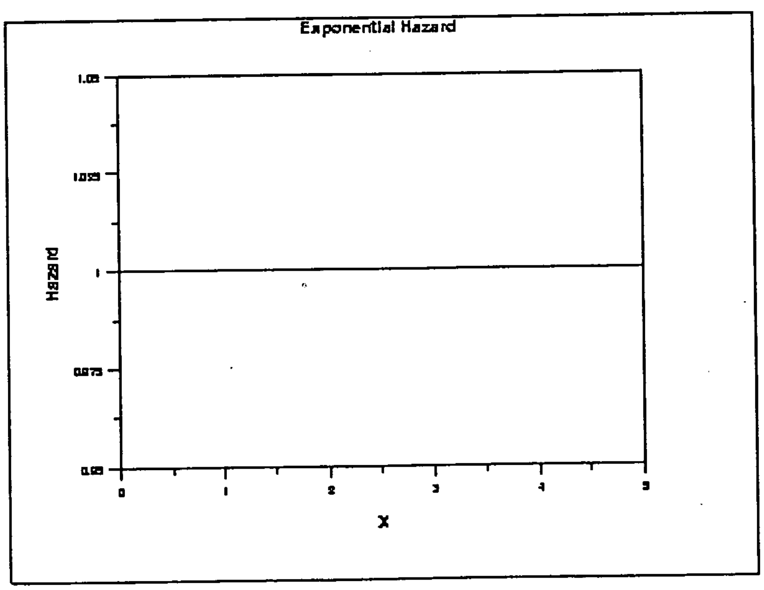


Hazard Function

The formula for the hazard function of the exponential distribution is

$$h(x) = \frac{1}{\beta} \qquad x \ge 0; \beta > 0$$

The following is the plot of the exponential hazard function.

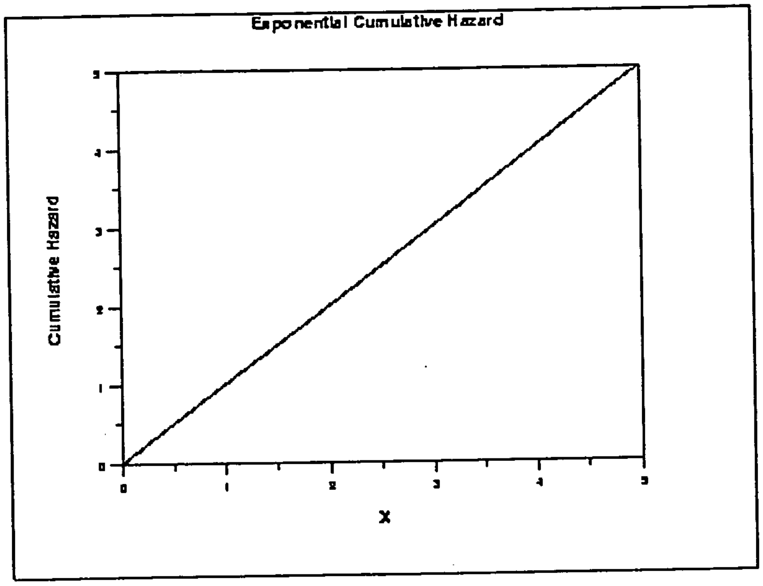


Cumulative Hazard Function

The formula for the cumulative hazard function of the exponential distribution is

$$H(x) = \frac{x}{\beta} \qquad x \geq 0; \beta > 0$$

The following is the plot of the exponential cumulative hazard function.

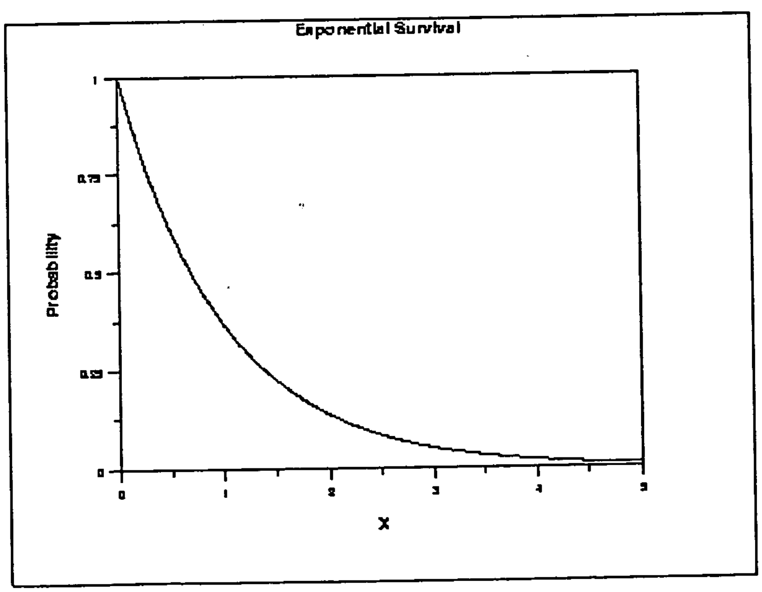


Survival Function

The formula for the survival function of the exponential distribution is

$$S(x) = e^{-x/\beta} \qquad x \geq 0; \beta > 0$$

The following is the plot of the exponential survival function.

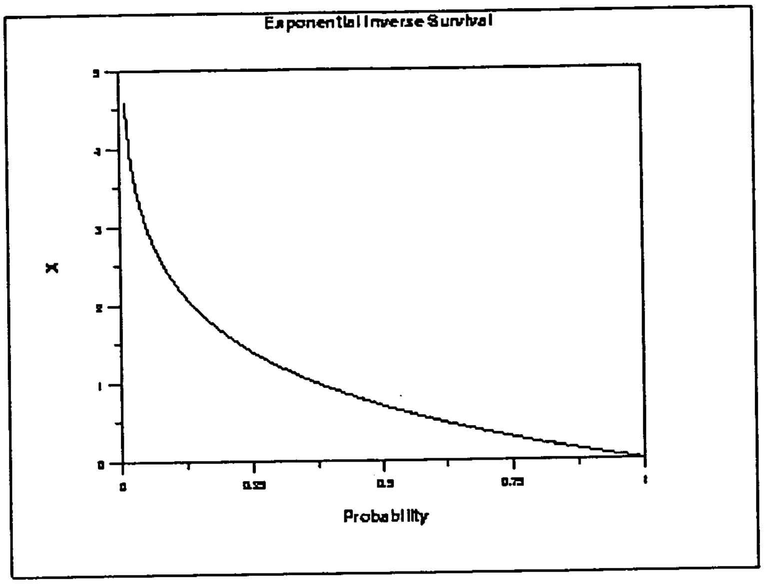


Inverse Survival Function

The formula for the inverse survival function of the exponential distribution is

$$Z(p) = -\beta \ln(p) \qquad 0 \le p < 1; \beta > 0$$

The following is the plot of the exponential inverse survival function.



Common Statistics

Mean	β
Median	$\beta \ln 2$
Mode	Zero
Range	Zero to plus infinity
Standard Deviation	β
Coefficient of Variation	1
Skewness	2
Kurtosis	9

Parameter Estimation

For the full sample case, the maximum likelihood estimator of the scale parameter is the sample mean. Maximum likelihood estimation for the exponential distribution is discussed in the chapter on reliability (Chapter 8). It is also discussed in chapter 19 of Johnson, Kotz, and Balakrishnan.

Comments

The exponential distribution is primarily used in reliability applications. The exponential distribution is used to model data with a constant failure rate (indicated by the hazard plot which is simply equal to a constant).

Software

Most general purpose statistical software programs, including Dataplot, support at least some of the probability functions for the exponential distribution.

1.3.6.6.8. Weibull Distribution

Probability Density Function

The formula for the probability density function of the general Weibull distribution is

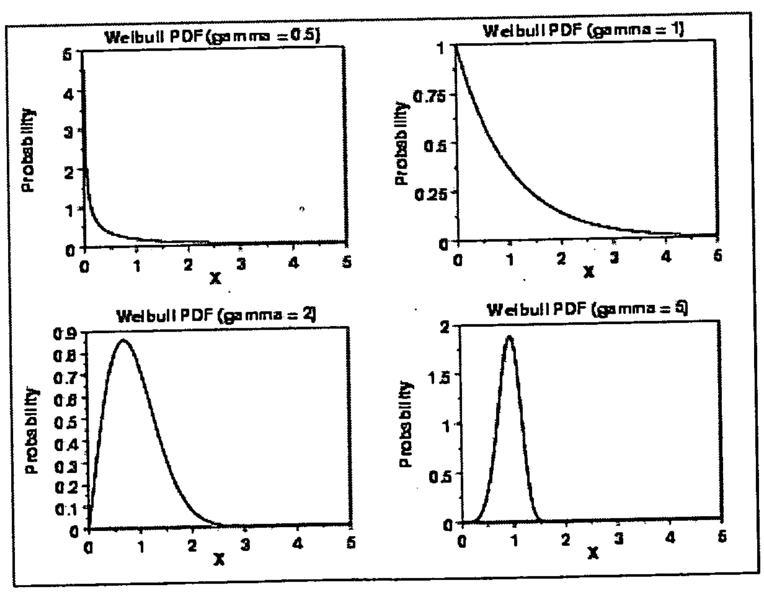
$$f(x) = \frac{\gamma}{\alpha}\left(\frac{x-\mu}{\alpha}\right)^{(\gamma-1)} \exp\left(-((x-\mu)/\alpha)^{\gamma}\right) \qquad x \ge \mu; \gamma, \alpha > 0$$

where γ is the shape parameter, μ is the location parameter and α is the scale parameter. The case where $\mu = 0$ and $\alpha = 1$ is called the **standard Weibull distribution**. The case where $\mu = 0$ is called the 2-parameter Weibull distribution. The equation for the standard Weibull distribution reduces to

$$f(x) = \gamma x^{(\gamma-1)} \exp(-(x^{\gamma})) \qquad x \ge 0; \gamma > 0$$

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the Weibull probability density function.



Cumulative

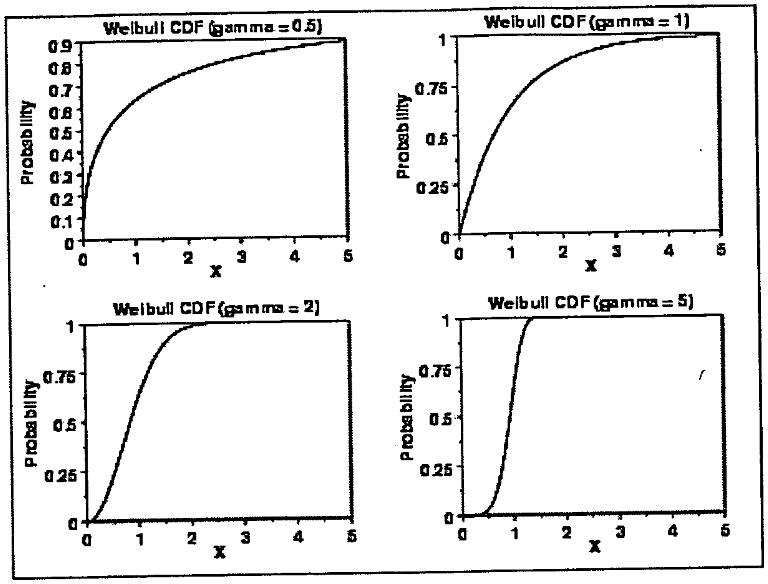
The formula for the cumulative distribution function of the Weibull

Distribution Function

distribution is

$$F(x) = 1 - e^{-(x^{\gamma})} \qquad x \ge 0; \gamma > 0$$

The following is the plot of the Weibull cumulative distribution function with the same values of γ as the pdf plots above.

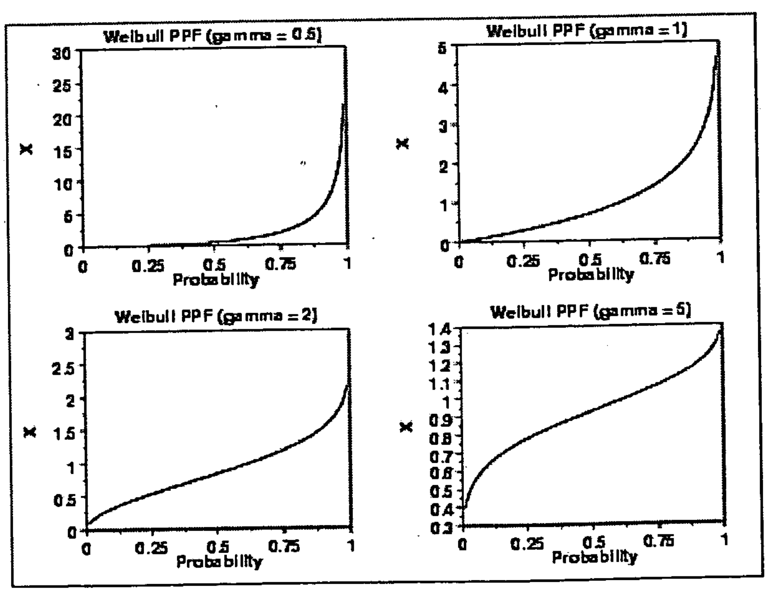


Percent Point Function

The formula for the percent point function of the Weibull distribution is

$$G(p) = (-\ln(1-p))^{1/\gamma} \qquad 0 \le p < 1; \gamma > 0$$

The following is the plot of the Weibull percent point function with the same values of γ as the pdf plots above.

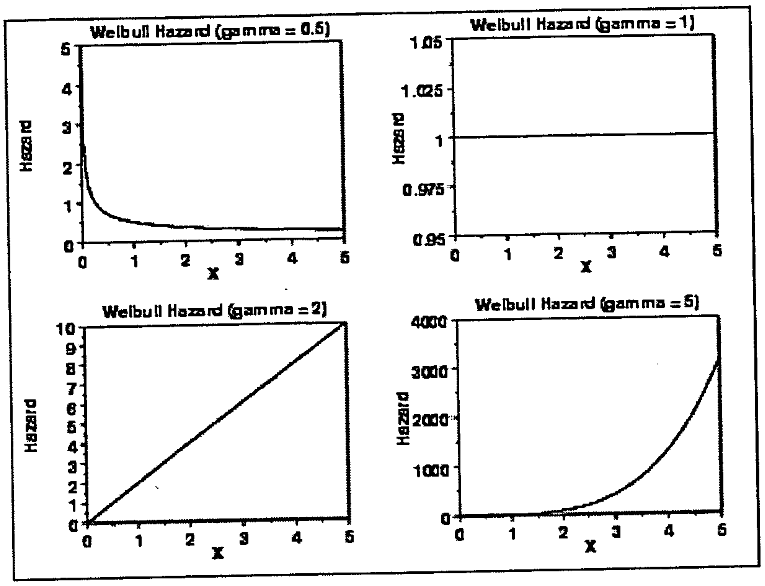


Hazard Function

The formula for the hazard function of the Weibull distribution is

$$h(x) = \gamma x^{(\gamma - 1)} \qquad x \ge 0; \gamma > 0$$

The following is the plot of the Weibull hazard function with the same values of γ as the pdf plots above.

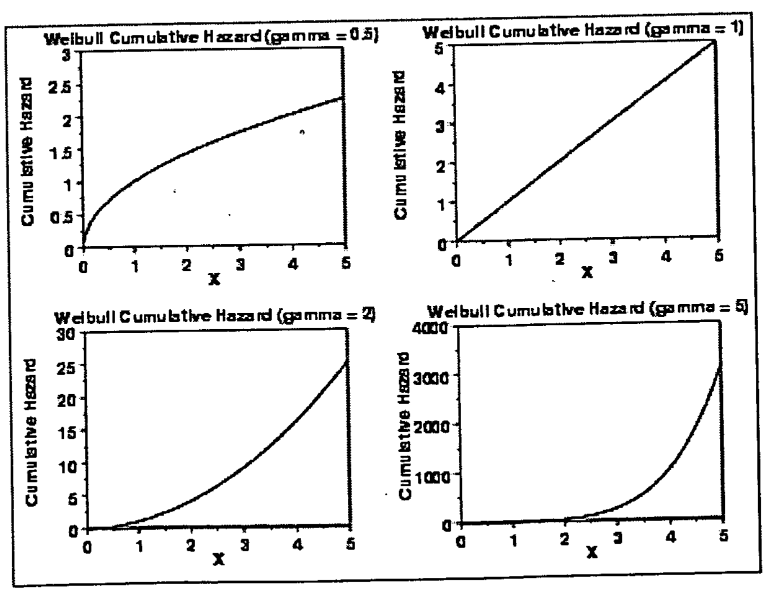


Cumulative Hazard Function

The formula for the cumulative hazard function of the Weibull distribution is

$$H(x) = x^{\gamma} \qquad x \ge 0; \gamma > 0$$

The following is the plot of the Weibull cumulative hazard function with the same values of γ as the pdf plots above.

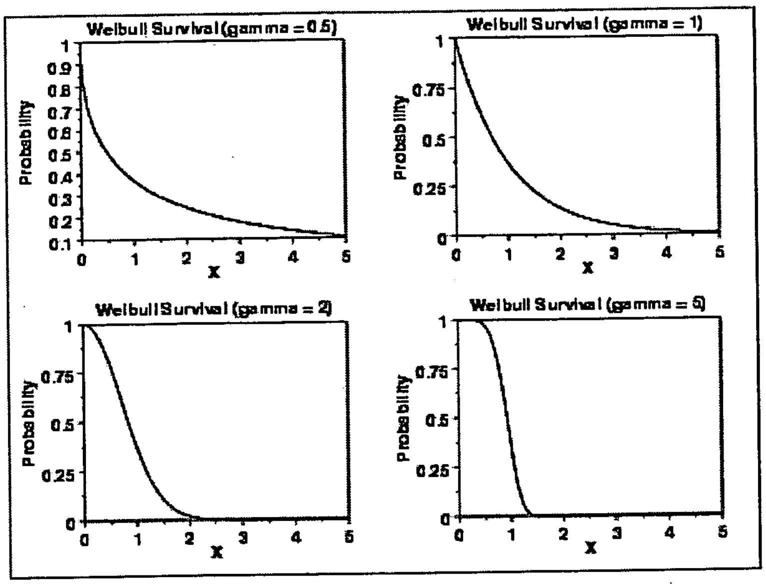


Survival Function

The formula for the survival function of the Weibull distribution is

$$S(x) = \exp -(x^{\gamma}) \qquad x \ge 0; \gamma > 0$$

The following is the plot of the Weibull survival function with the same values of γ as the pdf plots above.

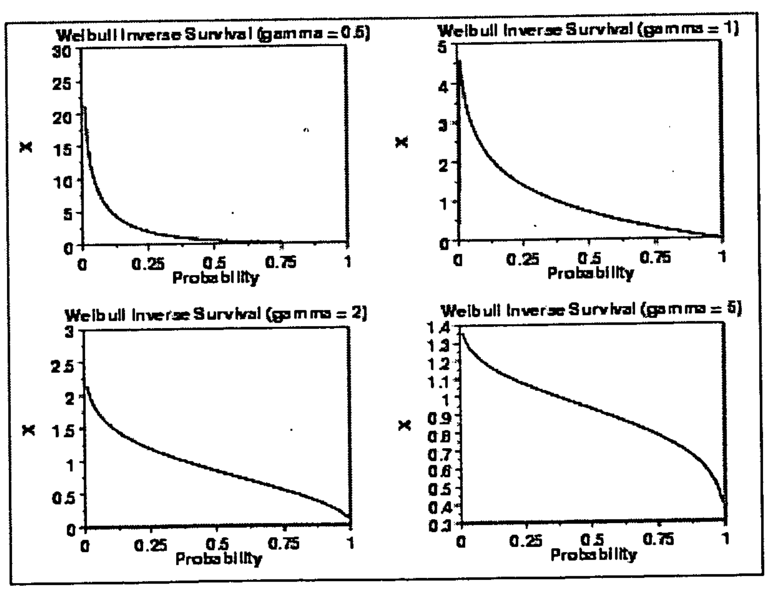


Inverse Survival Function

The formula for the inverse survival function of the Weibull distribution is

$$Z(p) = (-\ln(p))^{1/\gamma} \qquad 0 \le p < 1; \gamma > 0$$

The following is the plot of the Weibull inverse survival function with the same values of γ as the pdf plots above.



Common Statistics

The formulas below are with the location parameter equal to zero and the scale parameter equal to one.

Mean
$$\Gamma(\frac{\gamma+1}{\gamma})$$

where Γ is the gamma function

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$$

Median
$$\ln(2)^{1/\gamma}$$

Mode
$$(1-\frac{1}{\gamma})^{1/\gamma} \qquad \gamma > 1$$
$$0 \qquad \gamma \le 1$$

Range
Zero to positive infinity.

Standard Deviation
$$\sqrt{\Gamma(\frac{\gamma+2}{\gamma}) - (\Gamma(\frac{\gamma+1}{\gamma}))^2}$$

Coefficient of Variation
$$\sqrt{\frac{\Gamma(\frac{\gamma+2}{\gamma})}{(\Gamma(\frac{\gamma+1}{\gamma}))^2} - 1}$$

Parameter Estimation

Maximum likelihood estimation for the Weibull distribution is discussed in the Reliability chapter (Chapter 8). It is also discussed in Chapter 21 of Johnson, Kotz, and Balakrishnan.

Comments

The Weibull distribution is used extensively in reliability applications to model failure times.

Software

Most general purpose statistical software programs, including Dataplot, support at least some of the probability functions for the Weibull distribution.

1.3.6.6.9. Lognormal Distribution

Probability Density Function

A variable X is lognormally distributed if Y = LN(X) is normally distributed with "LN" denoting the natural logarithm. The general formula for the probability density function of the lognormal distribution is

$$f(x) = \frac{e^{-((\ln((x-\theta)/m))^2/(2\sigma^2))}}{(x-\theta)\sigma\sqrt{2\pi}} \qquad x \geq \theta; m, \sigma > 0$$

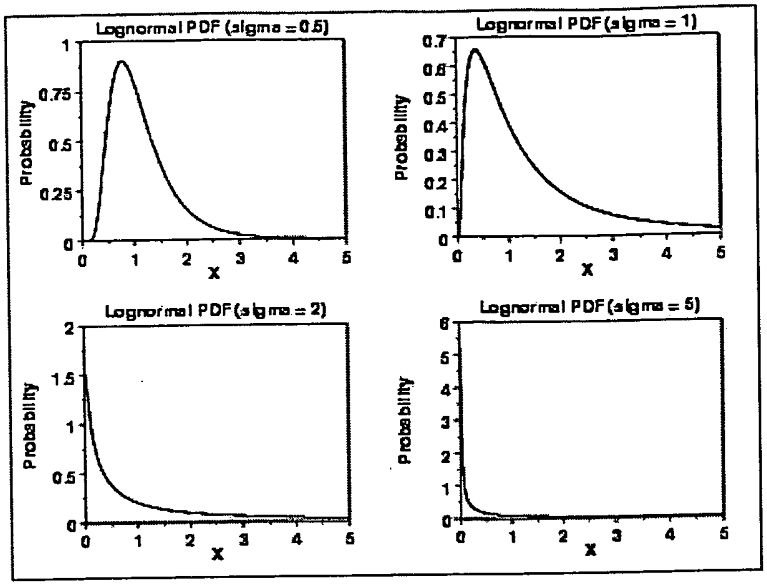
where σ is the shape parameter, θ is the location parameter and m is the scale parameter. The case where $\theta = 0$ and $m = 1$ is called the **standard lognormal distribution**. The case where θ equals zero is called the 2-parameter lognormal distribution.

The equation for the standard lognormal distribution is

$$f(x) = \frac{e^{-((\ln x)^2/2\sigma^2)}}{x\sigma\sqrt{2\pi}} \qquad x \geq 0; \sigma > 0$$

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the lognormal probability density function for four values of σ.



There are several common parameterizations of the lognormal distribution. The form given here is from Evans, Hastings, and Peacock.

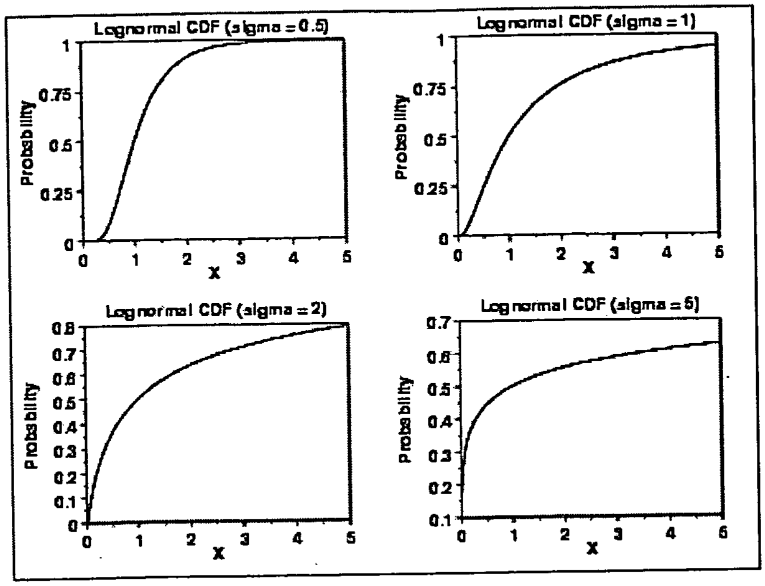
Cumulative Distribution Function

The formula for the cumulative distribution function of the lognormal distribution is

$$F(x) = \Phi(\frac{\ln(x)}{\sigma}) \qquad x \geq 0; \sigma > 0$$

where Φ is the cumulative distribution function of the normal distribution.

The following is the plot of the lognormal cumulative distribution function with the same values of σ as the pdf plots above.



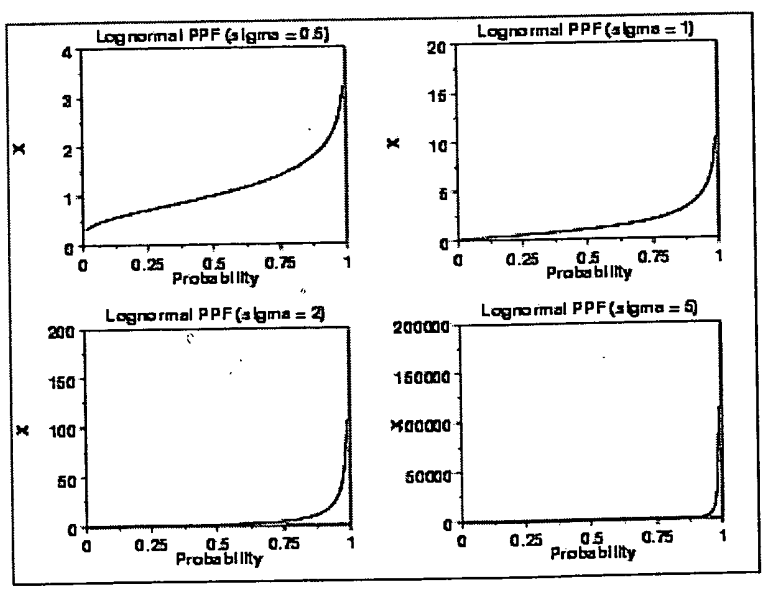
Percent Point Function

The formula for the percent point function of the lognormal distribution is

$$G(p) = \exp(\sigma\Phi^{-1}(p)) \qquad 0 \le p < 1; \sigma > 0$$

where Φ^{-1} is the percent point function of the normal distribution.

The following is the plot of the lognormal percent point function with the same values of σ as the pdf plots above.



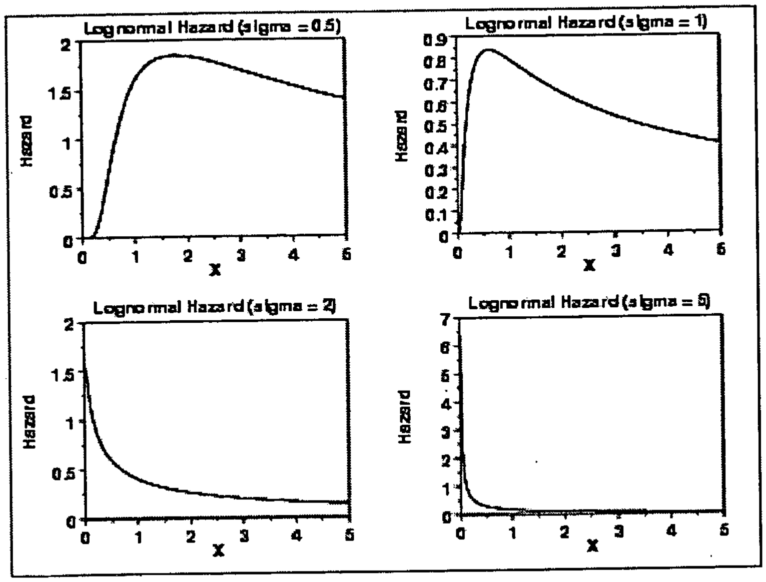
Hazard Function

The formula for the hazard function of the lognormal distribution is

$$h(x,\sigma) = \frac{(\frac{1}{x\sigma})\phi(\frac{\ln x}{\sigma})}{\Phi(\frac{-\ln x}{\sigma})} \qquad x > 0; \sigma > 0$$

where ϕ is the probability density function of the normal distribution and Φ is the cumulative distribution function of the normal distribution.

The following is the plot of the lognormal hazard function with the same values of σ as the pdf plots above.



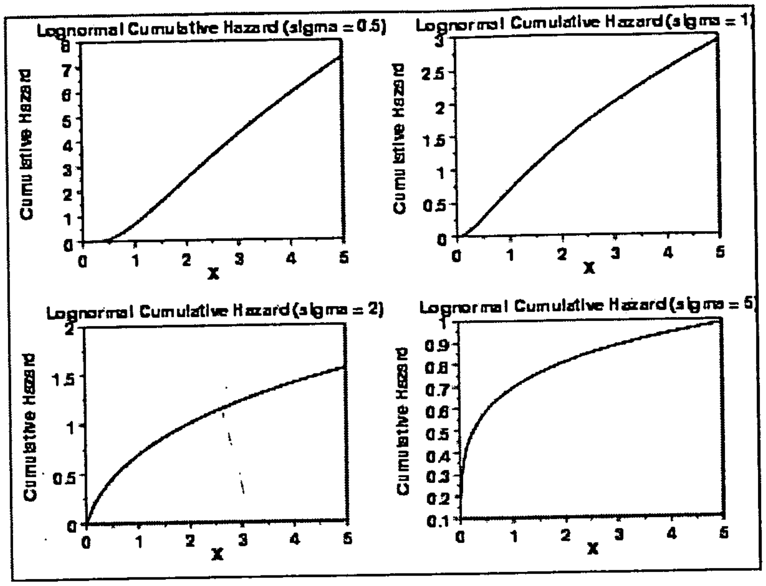
Cumulative Hazard Function

The formula for the cumulative hazard function of the lognormal distribution is

$$H(x) = \ln(1 - \Phi(\frac{\ln(x)}{\sigma})) \qquad x \geq 0; \sigma > 0$$

where Φ is the cumulative distribution function of the normal distribution.

The following is the plot of the lognormal cumulative hazard function with the same values of σ as the pdf plots above.



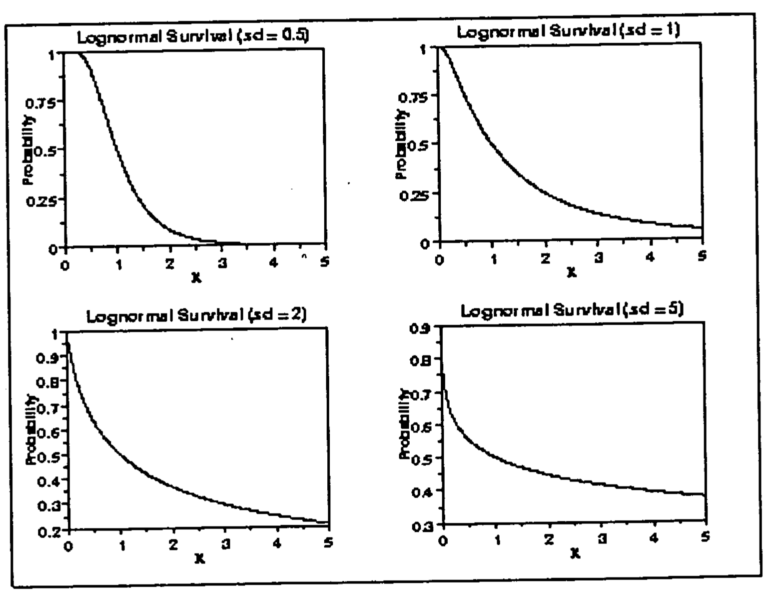
Survival Function

The formula for the survival function of the lognormal distribution is

$$S(x) = 1 - \Phi\left(\frac{\ln(x)}{\sigma}\right) \qquad x \ge 0; \sigma > 0$$

where Φ is the cumulative distribution function of the normal distribution.

The following is the plot of the lognormal survival function with the same values of σ as the pdf plots above.



Inverse Survival

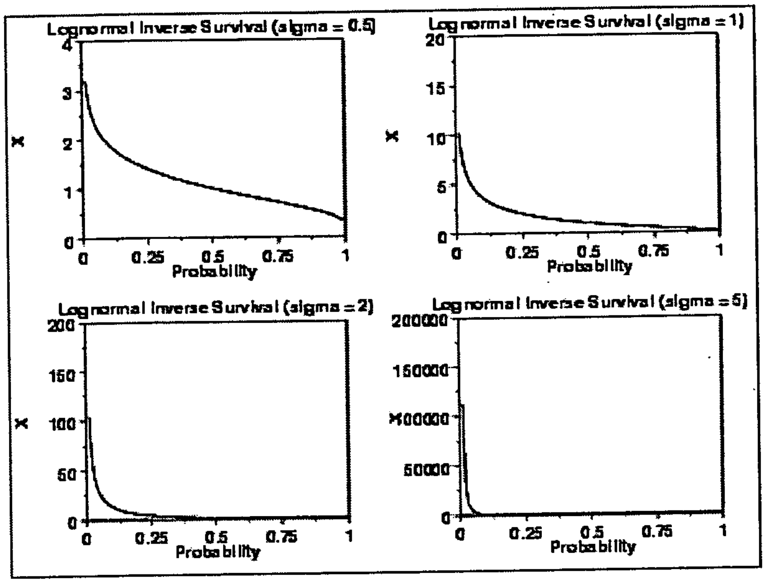
The formula for the inverse survival function of the lognormal distribution is

Function

$$Z(p) = \exp(\sigma\Phi^{-1}(1-p)) \qquad 0 \le p < 1; \sigma > 0$$

where Φ^{-1} is the percent point function of the normal distribution.

The following is the plot of the lognormal inverse survival function with the same values of σ as the pdf plots above.



Common Statistics

The formulas below are with the location parameter equal to zero and the scale parameter equal to one.

Mean	$e^{0.5\sigma^2}$
Median	Scale parameter ***m*** (= 1 if scale parameter not specified).
Mode	$\frac{1}{e^{\sigma^2}}$
Range	Zero to positive infinity
Standard Deviation	$\sqrt{e^{\sigma^2}(e^{\sigma^2}-1)}$
Skewness	$(e^{\sigma^2}+2)\sqrt{e^{\sigma^2}-1}$
Kurtosis	$(e^{\sigma^2})^4 + 2(e^{\sigma^2})^3 + 3(e^{\sigma^2})^2 - 3$
Coefficient of Variation	$\sqrt{e^{\sigma^2}-1}$

Parameter Estimation

The maximum likelihood estimates for the scale parameter, ***m***, and the shape parameter, σ, are

$$\hat{m} = \exp \hat{\mu}$$

and

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{N} (\ln(X_i) - \hat{\mu})^2}{N}}$$

where

$$\hat{\mu} = \frac{\sum_{i=1}^{N} \ln X_i}{N}$$

If the location parameter is known, it can be subtracted from the original data points before computing the maximum likelihood estimates of the shape and scale parameters.

Comments

The lognormal distribution is used extensively in reliability applications to model failure times. The lognormal and Weibull distributions are probably the most commonly used distributions in reliability applications.

Software

Most general purpose statistical software programs, including Dataplot, support at least some of the probability functions for the lognormal distribution.